On the Equilibria in Logit Models of Social Interaction and Quantal Response Equilibrium

John K. Dagsvik
Research Department, Statistics Norway and Department of Health Management and Health Economics, University of Oslo

UNIVERSITY OF OSLO
HEALTH ECONOMICS RESEARCH NETWORK

Working paper 2020:5

Health Economics Research Network at the University of Oslo
ISSN 1501-9071 (print version.), ISSN 1890-1735 (online), ISBN 978-82-7756-270-4

24 September 2020

On the Equilibria in Logit Models of Social Interaction and Quantal Response Equilibrium

by

John K. Dagsvik

Research Department, Statistics Norway, and
Department of Health Management and Health Economics, University of Oslo

Abstract

This paper investigates the set of equilibria in models of social interaction and Quantal Response Equilibrium (QRE). First, we discuss how models of social interaction can be viewed as a special case of QRE. Subsequently, we establish criteria that characterize the set of equilibria in models of social interaction and QRE. Finally, we establish conditions for convergence of sequential stochastic game models to QRE when players learn about the aggregate behavior of the players.

Key words: Random utility models, Behavioral game theory, Social interaction, Quantal Response Equilibrium

JEL classification: C02; C25; C62; C72; C73

Acknowledgement: I am grateful for discussions and comments by Ge Ge, Geir Godager, Steinar Strøm and Terje Skjerpen

1. Introduction

In standard textbook applications of game theory, players are assumed to behave perfectly rational and being able to account for other players' uncertain actions in a consistent (optimal) way when computing and maximizing (expected) payoffs. McKelvey and Palfrey (1995, 1998) extended the notion of Nash equilibrium in game theory to a corresponding stochastic theory denoted Quantal Response Equilibrium (QRE). Thus, in the QRE model, perfectly rational expectations equilibrium embodied in mixed strategy Nash equilibrium is replaced by an imperfect, or noisy, rational expectations equilibrium meaning that the players are assumed to maximize expected utility plus noise (Goeree et al. 2005, 2016). The QRE comprises a limiting case where the QRE coincides with a subset of Nash equilibria (Nash, 1950). Related approaches are discussed by Anderson et al. (2002) and Chen et al. (1997). Haile et al. (2008) have discussed the empirical content of QRE and Melo et al. (2019) have discussed testing of QRE models.

It is known that QRE models may have a single or several equilibria. Specifically, McKelvey and Palfrey (1995) proved that an equilibrium exists in QRE models. The purpose of this paper is to establish conditions for the existence of single and multiple equilibria in symmetric logit QRE models. We start by discussing logit choice models with social interaction which in some cases can be viewed as special cases of QRE models. In models with social interaction the preferences of an individual depend on the aggregate behavior of others. Choice models with social interaction allow the researcher to address how individual characteristics and aspects of social behavior interact, consistent with typical views in social science (Coleman, 1988, 1990). Models with social interactions have been applied to a wide variety of problems within social science, see for example Brock and Durlauf (2001), Kirman and Zimmermann (2001), and Kline and Tamer (2020). In the binary logit model with social interaction equilibrium conditions have been discussed by Becker (1974, 1991) and Brock and Durlauf (2001).

In this paper we show how the technique used to establish conditions for equilibrium in binary logit model with social interaction can be extended to the multinomial case. Subsequently, we extend our results to the case of symmetric QRE models. Thereafter, we consider the case with a population of pairs of players playing repeated games where the players ex ante are ignorant about the behavior of other players but are updated about the aggregate behavior of the players at each stage. In this case we establish conditions for the convergence to QRE.

The paper is organized as follows. In section 2 we present a general framework for the QRE model. Section 3 discusses a typical multinomial logit model of social interactions. In section 4 we analyze symmetric QRE models. Section 5 discusses the case of repeated games in a stationary environment.

2. Multinomial logit quantal response games

Consider a setting with two players, player *a* and player *b*. To player *a* there are *m* alternatives available and *n* alternatives are available to player *b*. Given that player *b* chooses alternative *k* then if player *a* chooses alternative *j* then player *a* receives payoff v_{jk}^{a}. Similarly, the payoff to player *b* is v_{kj}^{b} if player *b* chooses alternative *k* given that player *a* chooses alternative *j*. The payoff matrices $\{v_{jk}^{a}, v_{kj}^{b}\}$ are assumed to be common knowledge. Let, the probability that player *s* shall choose alternative *j* equals $P^{s}(j)$, $s = a, b$. The respective expected payoffs to player *a* and *b* when choosing alternatives *j* and *k* are therefore given by

$$\sum_{r=1}^{n} v_{jr}^{a} P^{b}(r) \quad \text{and} \quad \sum_{r=1}^{m} v_{kr}^{b} P^{a}(r)$$

The players are assumed to make choices which maximize U_{j}^{a} and U_{k}^{b}, respectively, given by

$$U_{j}^{a} = \sum_{r=1}^{n} v_{jr}^{a} P^{b}(r) + \varepsilon_{j}^{a} \quad \text{and} \quad U_{k}^{b} = \sum_{r=1}^{m} v_{kr}^{b} P^{a}(r) + \varepsilon_{k}^{b}$$

where ε_{j}^{a} and ε_{k}^{b} are random variables for all *j* and *k*.

Assumption 1

The random error terms $\{\varepsilon_{j}^{a}, \varepsilon_{k}^{b}\}$ *are mutually independent and independent of* $\{P^{a}(j), P^{b}(k)\}$ *with*

$$P(\varepsilon_{j}^{a} \le x) = \exp(-\exp(-x/\lambda^{a})) \quad \textit{and} \quad P(\varepsilon_{k}^{b} \le x) = \exp(-\exp(-x/\lambda^{b}))$$

for real x, where $\lambda_{j}^{a} > 0$ *and* $\lambda_{k}^{b} > 0$ *are constants.*

Assumption 1 means that the distributions of $\{\varepsilon_{j}^{a}, \varepsilon_{k}^{b}\}$ do not depend on the payoff matrix and that the behavior of the players satisfy probabilistic rationality (Luce, 1959).

Define $P^{a} = (P^{a}(1), P^{a}(2), \ldots)$, $V_{j}^{a}(P^{b}) = \lambda^{a} \sum_{r=1}^{n} v_{jr}^{a} P^{b}(r)$ and $V_{k}^{b}(P^{a}) = \lambda^{b} \sum_{r=1}^{m} v_{kr}^{b} P^{a}(r)$.

It follows from Assumption 1 that

(2.1) $$P^{a}(j) = P\left(U_{j}^{a} = \max_{s \le m} U_{s}^{a}\right) = \frac{\exp(V_{j}^{a}(P^{b}))}{\sum_{r=1}^{m} \exp(V_{r}^{a}(P^{b}))}$$

and

(2.2) $$P^b(k) = P\left(U_k^b = \max_s U_s^b\right) = \frac{\exp(V_k^b(P^a))}{\sum_{r=1}^{n}\exp(V_r^a(P^a))}$$

(McFadden, 1973). Under Assumption 1 the model given in (2.1) and (2.2) becomes the Quantal Response Equilibrium model (QRE). When $m=n$, $\lambda^a = \lambda^b = \lambda$ and $v_{jk}^a = v_{jk}^b = v_{jk} = v_{kj}$ the model reduces to a symmetric QRE with $P^a(j) = P^b(j) = P(j)$ where

(2.3) $$P(j) = \frac{\exp(V_j(P))}{\sum_{r=1}^{m}\exp(V_r(P))}$$

and

$$V_k(P) = \lambda \sum_{r=1}^{m} v_{kr} P(r).$$

We call this model the symmetric QRE model.

Proposition 1

The equilibrium choice probabilities given in (2.1) and (2.2) satisfy the following inequalities:

$$\frac{\exp(\min_s \lambda^a v_{js}^a)}{\exp(\min_s \lambda^a v_{js}^a) + \sum_{r\neq j}\exp(\max_s \lambda^a v_{rs}^a)} \leq P^a(j) \leq \frac{\exp(\max_s \lambda^a v_{js}^a)}{\exp(\max_s \lambda^a v_{js}^a) + \sum_{r\neq j}\exp(\min_s \lambda^a v_{rs}^a)}$$

and

$$\frac{\exp(\min_s \lambda^b v_{sk}^b)}{\exp(\min_s \lambda^b v_{sk}^b) + \sum_{r\neq k}\exp(\max_s \lambda^b v_{sr}^b)} \leq P^b(k) \leq \frac{\exp(\max_s \lambda^b v_{sk}^b)}{\exp(\max_s \lambda^b v_{sk}^b) + \sum_{r\neq k}\exp(\min_s \lambda^b v_{sr}^b)}.$$

The proof of Proposition 1 is given in the appendix.

The stochastic formulation of game theory enables researchers to formulate and estimate models in cases where the payoffs are unobservable utilities that may depend on several observable attributes, pecuniary as well as non-pecuniary ones. For example, in the symmetric case $(v_{jk} = v_{kj})$ where the normalized payoffs are given by $\lambda v_{jk} = Z_{jk}\beta$, $Z_{jk} = (Z_{jk}(1), Z_{jk}(2),...)$, is a vector of observable attributes and β is an unknown parameter vector to be estimated, the QRE model implies that

$$\log\left(\frac{P(j)}{P(m)}\right) = \sum_k P(k)(Z_{jk} - Z_{mk})\beta$$

We note that when the dimension of β is less than or equal to $m-1$ then β is identified provided the matrix $\{a_{jr}\}$ has rank $m-1$, where

$$a_{jr} = \sum_k P(k)(Z_{jk}(r) - Z_{mk}(r)).$$

3. Social interaction as a special case of QRE

Some authors, such as Becker (1974, 1991), Becker and Murphy (2000), Brock and Durlauf (2001, 2002), Kirman and Zimmermann (2001), Manski (2000) and Shelling (1971), have discussed different settings with social interaction in economics. We shall now see how choice behavior when the preferences are influenced by the aggregate behavior of others, can be formulated as a QRE model.

Let S_i be the set of individuals in the peer group of agent *I*, and m_i the number of individuals in S_i. Let $\xi_{kj} = 1$ if alternative *j* is the most preferred alternative of individual *k* in the population and equal to zero otherwise. The individuals in S_i are perceived as being well informed by agent *i* and their judgments are trusted by the agent. The aggregate behavior of individuals in S_i is known to her or him. The utility of agent *i* of alternative *j* is assumed to have the following structure:

(3.1) $$U_{ij} = \alpha_j + 0.5\beta \sum_{k \in S_i} \xi_{kj} / m_i - 0.5\beta \sum_{k \in S_i} (1 - \xi_{kj}) / m_i + \breve{\varepsilon}_{ij}$$

where α_j is a deterministic term that is known to the agent and β is a non-negative parameter. Thus, $\alpha_j + 0.5\beta$ is the utility of alternative *j* of agent *i* if alternative *j* is the most preferred alternative of agent *k* and equal to $\alpha_j - 0.5\beta$ if alternative *j* is not the most preferred alternative of agent *k*. The variable ξ_{kj} is uncertain to agent *i*. The variable $\breve{\varepsilon}_{ij}$ is a stochastic error term. The error terms $\{\tilde{\varepsilon}_{ij}\}$ may fluctuate from one moment to the next due to the inability of the agent to assess the precise value (to him) of the alternatives. Since all the terms $\{\alpha_k\}$ cannot be identified there is no loss of generality by letting $\alpha_m = 0.$ Let $\breve{P}_{ij} =: E^s \xi_{kj}$ for $k \in S_i$, where E^s denotes the subjective expectation operator. That is, $\breve{P}_i(j)$ is the perceived subjective probability that agent *k* in the agent's peer group shall choose alternative *j*. Using (3.1) it follows that the subjective expected utility of agent *i* is equal to

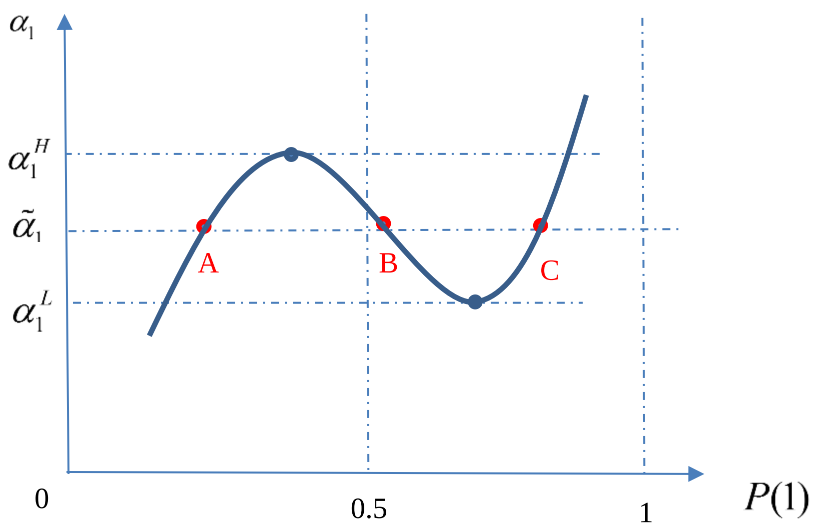
(3.2) $$E^s U_{ij} = \alpha_j + \beta \breve{P}_i(j) + \breve{\varepsilon}_{ij}.$$

Let $P_j = E\breve{P}_{ij} = EE^s\xi_{kj}$ for $k \in S_i$ where E is the objective (population) expectation operator. We can rewrite (3.2) as

(3.3) $$E^s U_{ij} = \alpha_j + \beta P(j) + \beta(\breve{P}(j)_i - P(j)) + \breve{\varepsilon}_{ij} = \alpha_j + \beta P(j) + \varepsilon_{ij}$$

where $\varepsilon_{ij} = \beta(\breve{P}_i(j) - P(j)) + \breve{\varepsilon}_{ij}$. If $\varepsilon_{ij}, j = 1,2,...,m,$ are assumed independent and distributed according to Assumption 1 with $\lambda = 1,$ it follows that

Figure 1. Equilibria in the binary choice model



(3.4) $$P(j) = \frac{\exp(\alpha_j + \beta P(j))}{\sum_{r=1}^{m} \exp(\alpha_r + \beta P(r))}$$

We note that the structure of the formula in (3.4) is a special case of the framework considered above with $v_{jk} = \alpha_j + \beta\delta_{jk}$.

In the binary case the model in (3.4), with the normalization $\alpha_2 = 0,$ reduces to

(3.5) $$P(1) = \frac{\exp(\alpha_j + \beta P(1))}{\sum_{r=1}^{2} \exp(\alpha_r + \beta P(r))} = \frac{1}{1 + \exp(-\alpha_1 - 2\beta P(1))}.$$

Becker (1991) has discussed the model in (3.5). It may be instructive to repeat part of his discussion here. To this end suppose that α_1 is a function of a variable such as for example the price difference between the two alternatives. In this case it follows that the “inverse aggregate demand” equals

(3.6) $$\alpha_1 = \beta + \log\left(\frac{P(1)}{1-P(1)}\right) - 2\beta P(1)$$

which implies that

(3.7) $$\frac{\partial \alpha_1}{\partial P(1)} = \frac{1}{(1-P(1))P(1)} - 2\beta.$$

Since $(1-P(1))P(1) \leq 1/4$ it follows from (3.7) that the function (3.6) has the shape as given in Figure 1 provided $\beta > 2.$ Thus, we realized that if $\alpha_1^L < \alpha_1 < \alpha_1^H$ and $\beta > 2$ three equilibria exists. However, only the equilibria *A* and *C* are stable whereas *B* is unstable.

In the general case with *m* alternatives Brock and Durlauf (2002) have proved that when α_j is independent of *j* for all *j* then the model in (3.4) has multiple equilibria when $\beta > m.$ The next result extends the result of Brock and Durlauf (2002) to the case with general $\{\alpha_j\}.$ Before stating the result we need the following notation. Let

(3.8) $$f(x) = 0.5\sqrt{x(x-4)} - \log\left(0.5x - 1 + 0.5\sqrt{x(x-4)}\right)$$

for $x \geq 4.$ It follows readily from (3.8) that $f(x)$ is strictly increasing for $x > 4$. Since $f(4) = 0$ it follows that $f(x) > 0$ when $x > 4.$

Theorem 1

Let $\tilde{P}$ be any stable equilibrium vector of choice probabilities satisfying (3.4) and let $\theta = \tilde{P}(1) + \tilde{P}(2).$

(i) If $\beta\theta < 2$

or

$\beta\theta > 2$ *and* $f(2\beta\theta) < |\alpha_1 - \alpha_2|$

for every enumeration of the alternatives then the equilibrium vector $\tilde{P}$ is unique.

(ii) If $\beta\theta > 2$ *and* $f(2\beta\theta) > |\alpha_1 - \alpha_2|$

there exist two other equilibria of which one is stable and the other one is unstable.

The proof of Theorem 1 is given in the appendix. Evidently, the enumeration of the alternatives is irrelevant for the result of the theorem. Thus, if $\theta = \tilde{P}(r) + \tilde{P}(s)$ where $r \neq s$ then the result in Theorem 1 holds with the obvious modification the indexation of the alfas. The next result follows from Theorem 1 and Proposition 1.

Corollary 1

If

$$\frac{\beta(\exp(\alpha_r)+\exp(\alpha_s))}{\exp(\alpha_r)+\exp(\alpha_s)+\exp(-\beta)\sum_{k\neq r,s}\exp(\alpha_k)}\leq 2$$

for all feasible combinations of r and s there is only one equilibrium of the choice probabilities.

4. Equilibria in the symmetric QRE model

For simplicity we start by considering the symmetric binary case with two alternatives available to both players. In this case the QRE model be written as

(4.1) $$P(1)=\frac{\exp(v_{11}P(1)+v_{12}P(2))}{\sum_{r=1}^{2}\exp(v_{r1}P(1)+v_{r2}P(2))}=\frac{1}{1+\exp(-(v_{11}+v_{22}-2v_{12})P(1)-v_{12}+v_{22}))}.$$

Since the model given in (4.1) has the same mathematical structure as the model in (3.4) the next result follows.

Theorem 2

Assume a symmetric game with m =2.

(i) *If either* $(v_{11}-2v_{12}+v_{22})<4$,

or

$$(v_{11}-2v_{12}+v_{22})>4 \ \textit{and} \ f(v_{11}-2v_{12}+v_{22})<0.5|v_{11}-v_{22}|$$

then there exists a unique equilibrium.

(ii) *If* $(v_{11}-2v_{12}+v_{22})>4$ *and* $f(v_{11}-2v_{12}+v_{22})>0.5|v_{11}-v_{22}|$

there exist 3 equilibria of which two are stable and one is unstable.

The proof of Theorem 2 is given in the appendix. McKelvey and Palfrey (1995) have proved that there exists at least one equilibrium in the general QRE model, that the center of the simplex is an equilibrium when $\lambda \to 0$, and that "for almost all games there is a unique selection when $\lambda \to \infty$ (McKelvey and Palfrey, 1995 p. 12.).

Next, we shall apply a variant of the approach used to prove Theorem 2 to obtain similar results for the multinomial case. In this case the equation system that defines equilibrium is given by

(4.2) $$P(j)=\frac{\exp(V_j(P))}{\sum_{r=1}^{m}\exp(V_r(P))}, \qquad V_j(P)=\sum v_{jk}P(k)$$

for j = 1, 2, …, m. Let $\tilde{P}$ be any given equilibrium vector (which exists).

Theorem 3

Assume that the payoff matrix $\{v_{jk}\}$ *is symmetric. Let* $\tilde{P}$ *be any equilibrium vector of choice probabilities and let* $\theta=\tilde{P}(1)+\tilde{P}(2)$ *and* $q=P(1)/\theta$.

(i) If either $(v_{11}-2v_{12}+v_{22})\theta<4$,

or

$$(v_{11}-2v_{12}+v_{22})\theta>4 \text{ and } f((v_{11}+v_{22}-2v_{12})\theta)<|0.5(v_{11}-v_{22})\theta+\sum_{r=3}^{m}(v_{1r}-v_{2r})\tilde{P}(r)|$$

for every enumeration of the alternatives, then the equilibrium vector $\tilde{P}$ *is unique.*

(ii) If $(v_{11}-2v_{12}+v_{22})\theta>4$ *and* $f((v_{11}+v_{22}-2v_{12})\theta)>|0.5(v_{11}-v_{22})\theta+\sum_{r=3}^{m}(v_{1r}-v_{2r})\tilde{P}(r)|$

there exist two other equilibria for q (for given $\tilde{P}$*) of which one is stable and the other one is unstable.*

The results obtained in Theorem 3 provide necessary and sufficient conditions for unique- or multiple equilibria in the symmetric QRE model. As noted above, the enumeration of the alternatives is irrelevant for the result of the theorem. If m = 3 for example, there are 3 possible enumerations of the alternatives, so that either $\theta=\tilde{P}(1)+\tilde{P}(2)$ or $\theta=\tilde{P}(1)+\tilde{P}(3)$.

5. Equilibrium choice probabilities obtained by repeated stationary games

The QRE model is implicitly based on the assumption that the players are somehow able to calculate the equilibrium choice probabilities. However, no theory is offered on how the players actually are able to find the equilibrium probabilities. In this section we consider the case of repeated games in a stationary environment. By stationary environment it is understood that the payoff matrices of the players remain unchanged over the repetitions. We show below that if players play repeated games in a stationary environment then, under specific conditions, the corresponding iterative process will converge to equilibrium.

Consider a large population of observationally identical players. Each player makes his choice based upon the maximization of a linear combination of expected utility and a stochastic error term, given the population choices of other players. A priori, the players are assumed ignorant about the equilibrium choice probabilities. The players play repeated games in several stages in a stationary environment. In the first stage the players select an arbitrary vector of choice probabilities which is used to calculate the systematic term of the respective expected utility or payoff. In stage two the game is played where the observed relative frequencies in the population obtained from the first stage (which are assumed to be common knowledge) are used to get an updated estimate of the expected payoffs. Provided the population is large, the observed fractions of choices will be close to the corresponding theoretical choice probabilities (stage dependent), since relative frequencies are consistent estimators of the choice probabilities. In stage three, the game is played again based on the updated expected payoffs, and so on, until convergence is achieved. To state the next result we need some additional notation.

Define

$$F_j^a(P^b) = \frac{\exp(V_j^a(P^b))}{\sum_{r\le m} \exp(V_r^a(P^b))} \quad \text{and} \quad F_k^b(P^a) = \frac{\exp(V_k^b(P^a))}{\sum_{r\le n} \exp(V_r^b(P^a))}$$

where P^t, $t = a, b$, are the vectors of choice probabilities. Let

$$\Delta^a = \{P : P(j) \in (0,1), j \le m, \textstyle\sum_{r=1}^m P(r) = 1\}, \quad \Delta^b = \{P : P(j) \in (0,1), j \le n, \textstyle\sum_{r=1}^n P(r) = 1\},$$

$F^a(P^b) = (F_1^a(P^b), F_2^a(P^b), \ldots)$, and let $F^b(P^a)$ be define similarly.

Theorem 4

Assume that the expected payoff matrix satisfies the following inequalities

$$\max_j(\max_k(v_{jk}^a - v_{mk}^a) - \min_k(v_{jk}^a - v_{mk}^a)) < \frac{n}{n-1}$$

and

$$\max_k(\max_j(v_{kj}^b - v_{nj}^b) - \min_j(v_{kj}^b - v_{nj}^b)) < \frac{m}{m-1}.$$

Then the mapping $(P^a, P^b) \to (F^a(P^b), F^b(P^a))$ *is a contraction and therefore has a unique fixed point on* (Δ^a, Δ^b).

The proof of Theorem 4 is given in the appendix. Theorem 4 states that under the specific condition on the payoff matrix there exists a unique equilibrium value of the respective systematic terms of the expected utilities.

The next Corollary follows readily.

Corollary 2

Assume that n = m, $v_{jk}^{a} = v_{kj}^{b} = v_{jk} = v_{kj}$. *Then the mapping* $P^a = P^b = P \rightarrow F(P)$ *is a contraction and therefore the QRE exists and is unique provided*

$$\max_j (\max_k v_{jk} - \min_k v_{jk}) < \frac{m}{m-1}.$$

6. Conclusions

In this paper we have developed conditions for determining the number of stable equilibria in QRE models. Specifically, we have demonstrated how the idea used to analyze the set of equilibria in the binary model of social interaction can be extended to the multinomial case and more generally, to symmetric QRE models. Under stronger conditions we have shown that repeated games (not necessarily symmetric) in a stationary environment where the players do not know the expected payoffs will converge to QRE.

Funding: This study was funded by the Research Department of Statistics Norway and the Ragnar Frisch Centre of Economic Research, Oslo

Conflict of interest: The author declares that he has no conflict of interest

References

Anderson, S. P., J. K. Goeree and C. A. Holt (2002): The Logit Equilibrium. A Perspective on Intuitive Behavioral Anomalies. *Southern Economic Journal*, **69**, 21-47.

Becker, G. S. (1974): A Theory of Social Interactions. *Journal of Political Economy*, **82**, 1063-1093.

Becker, G. S. (1991): A Note on Restaurant Pricing and Other Examples of Social Influences on Price. *Journal of Political Economy,* **99**, 1109-1116.

Becker, G. S. and K. Murphy (2000*): Social Economics: Market Behavior in a Social Environment*. Harvard University Press, Cambridge, MA.

Brock, W. A. and S. N. Durlauf (2001): Discrete Choice with Social Interactions. *Review of Economic Studies*, **68**, 235-260.

Brock, W. A. and S. N. Durlauf (2002): Multinomial-Choice Model of Neighborhood Effects. *The American Economic Review*, **92**, 298-303.

Brock, W. A. and S. N. Durlauf (2003): A Multinomial Choice with Social Interaction. Mimeo, Department of Economics, University of Wisconsin.

Brock, W. A. and S. N. Durlauf (2001): Interactions-based models. In J. J. Heckman and E Leader (eds.), *Handbook of Econometrics*, Vol 5, Elsevier, New York.

Coleman, J. (1988): Social capital in the creation of human capital. *American Journal of Sociology*, **94** (supplement), S95-S120.

Coleman, J. (1990): *Foundations of Social Theory*. Harvard University Press, Cambridge.

Chen, Hsiao-Chi, J. W. Friedman and J.-F. Thisse (1997): Boundedly Rational Nash Equilibrium: A Probabilistic Choice Approach. *Games and Economic Behavior*, **18**, 32-54.

Goeree, J. K., C. A. Holt and T. R. Palfrey (2005): Regular Quantal Response Equilibrium. *Experimental Economics*, **8**, 347-67.

Goeree, J. K., C. A. Holt and T. R. Palfrey (2016): *Quantal Response Equilibrium: A Stochastic Theory of Games.* Princeton University Press.

Haile, P. A., A. Hortacsu and G. Kosenok (2008): On the Empirical Content of Quantal Response Equilibrium. *American Economic Review*, **98**, 180-200.

Kirman, A., J.-B. Zimmermann (eds.) (2001): *Economics with Heterogeneous Interacting Agents.* Lecture Notes in Economics and Mathematical Systems. Springer Verlag, Berlin.

Kline, B. and E. Tamer (2020): *Econometric Analysis of Models with Social Interactions. The Econometric Analysis of Network Data*. Academic Press, New York.

Luce, R. D. (1959): *Individual Choice Behavior: A Theoretical Analysis*. Wiley, New York,

Manski, C. F. (2000): Economic Analysis of Social Interactions. *Journal of Economic Perspectives*, **14**, 115-136.

Melo, E., K. Pogorelskiy and M. Shum (2019): Testing the quantal response hypothesis. *International Economic Review*, **60**, 53-74.

McKelvey, R. D. and T. R. Palfrey (1995): Quantal Response Equilibria in Normal Form Games. *Games and Economic Behavior,* **10**, 6-38.

McKelvey, R. D. and T. R. Palfrey (1998): Quantal Response Equilibria for Extensive Games. *Experimental Economics,* **1**, 9-41.

McFadden, D. (1973): Conditional Logit Analysis of Qualitative Choice Behavior. In P. Zarembka (ed.), *Frontiers in Econometrics*, Academic Press, New York.

Nash, J. F. Jr, (1950): Equilibrium Points in n-Person Games. *Proceedings of the National Academy of Sciences*, **36**, 48-49.

Rudin, W. (1976): *Principles of Mathematical Analysis*, 3nd edition. McGraw-Hill Companies, Inc., New York.

Schelling, T. C. (1971): Dynamic Models of Segregation. *Journal of Mathematical Sociology*, **1**, 143-186.

Appendix

Proof of Proposition 1:

Evidently it must be true that $\min_s v_{js} \le V_j^a(P^b) \le \max_s v_{js}$. Remember that the function

$$\frac{\exp(y_j)}{\sum_k \exp(y_k)}$$

is increasing in y_j and decreasing in y_k for $k \neq j$. In equilibrium one must therefore have that

$$P^a(j) = \frac{\exp(V_j^a(P^b))}{\sum_{r\in X^a} \exp(V_r^a(P^b))} \le \frac{\exp(\max_s \lambda^a v_{js}^a)}{\exp(\max_s \lambda^a v_{js}^a) + \sum_{r\neq j} \exp(\min_s \lambda^a v_{rs}^a)}.$$

Similarly, we have that

$$P^a(j) = \frac{\exp(V_j^a(P^b))}{\sum_{r\in X^a} \exp(V_r^a(P^b))} \ge \frac{\exp(\min_s \lambda^a v_{js}^a)}{\exp(\min_s \lambda^a v_{js}^a) + \sum_{r\neq j} \exp(\max_s \lambda^a v_{rs}^a)}.$$

The corresponding inequalities for $P^b(k)$ are proved in a similar way.

Q.E.D.

Proof of Theorem 2:

Let $D = v_{11} + v_{22} - 2v_{12}$,

$$g(x) \equiv \log\left(\frac{x}{1-x}\right) - D(x-0.5), \text{ for } x \in (0,1),$$

and let x_1 and x_2 be the roots of $g'(x)$ determined by

$$g'(x_j) = \frac{1}{x_j(1-x_j)} - D = 0,$$

which are given by

$$x_1 = 0.5 - 0.5\sqrt{1-\frac{4}{D}} \quad \text{and} \quad x_2 = 0.5 + 0.5\sqrt{1-\frac{4}{D}}.$$

Note first that (4.1) is equivalent to

(A.1) $\qquad g(P(1)) = 0.5(v_{11} - v_{22}).$

We have that $g'(x) > 0$ if $D < 4$ because $x(1-x) \le 4$ for all $x \in (0,1).$ Thus, in this case only a single equilibrium can occur. If $D > 4$ and $g(P_1) > g(x_1)$ then also only one equilibrium can occur. Since $g(P_1) = (v_{11} - v_{22})/2$ the latter inequality is equivalent to $(v_{11} - v_{22})/2 > g(x_1).$ Similarly, if $D > 4$ and $(v_{11} - v_{22})/2 = g(P_1) < g(x_2)$ only one equilibrium can occur. If, however, $D > 4$ and $g(x_2) < (v_{11} - v_{22})/2 < g(x_1),$ we realize that 3 equilibria may occur, and it is easy to show that one of them is unstable. Note that

$$g(x_2) = \log\left(\frac{1+\sqrt{1-4/D}}{1-\sqrt{1-4/D}}\right) - 0.5D\sqrt{1-4/D}$$

$$= \log\left(0.5D - 1 + 0.5D\sqrt{1-4/D}\right) - 0.5D\sqrt{1-4/D}$$

$$= \log\left(0.5D - 1 + 0.5\sqrt{D(D-4)}\right) - 0.5\sqrt{D(D-4)} = -f(\kappa).$$

Similarly, it follows that $g(x_1) = f(D).$ Hence, the inequality $0.5(v_{11} - v_{22}) > g(x_1)$ is equivalent to $0.5(v_{11} - v_{22}) > f(D)$ and the inequality $0.5(v_{11} - v_{22}) < g(x_2)$ is equivalent to $0.5(v_{11} - v_{22}) < -f(D).$ Thus, the two inequalities can be expressed as $0.5|v_{11} - v_{22}| > f(D),$ which proves (i). The proof of (ii) is similar.

Q.E.D.

Proof of Theorem 3:

Let $D = v_{11} + v_{22} - 2v_{12}$ (as above) and

$$\psi(x) \equiv \log\left(\frac{x}{1-x}\right) - D\theta(x - 0.5),$$

for $x \in (0,1).$ (Note that when $\theta = 1$ then $\psi(x) = g(x).$) Moreover

$$\psi'(x) \equiv \frac{1}{x(1-x)} - D\theta.$$

Let z_1 and z_2 be the roots of $\psi'(x),$ that is, $\psi'(z_1) = \psi'(z_2) = 0.$ These roots are given by

(A.2) $$z_1 = 0.5 - 0.5\sqrt{1 - \frac{4}{D\theta}}, \quad z_2 = 0.5 + 0.5\sqrt{1 - \frac{4}{D\theta}}.$$

By using the results of Theorem 2 as a starting point it is easy to see how it can be extended to the general multinomial case. Note that whereas θ is given, $q = P(1)/\theta$ is so far not determined. The relation that determines *q*, and is equivalent to (4.2), is

(A.3) $$\psi(q) = 0.5(v_{11} - v_{22})\theta + \sum_{r=3}^{m}(v_{1r} - v_{2r})\tilde{P}(r).$$

Similarly to the proof of Theorem 2 it follows, with *f* given in (3.8), that

$$\psi(z_2) = -f(D\theta) \text{ and } \psi(z_1) = f(D\theta).$$

By using (A.2) and (A.3) we therefore get the next result which is analogous to the result of Theorem 1.

Q. E. D.

Proof of Theorem 1:

As in Theorem 2, define $\theta = P(1) + P(2),\ \ q = P(1)/\theta,$ and

$$\psi(x) = \log\left(\frac{x}{1-x}\right) - 2\beta\theta(x-0.5),$$

$$y_1 = 0.5 - 0.5\sqrt{1-\frac{2}{\beta\theta}} \quad \text{and} \quad y_2 = 0.5 + 0.5\sqrt{1-\frac{2}{\beta\theta}}\ .$$

It follows from (3.4) that the equilibrium probabilities must satisfy

$$\psi(q) = \alpha_1 - \alpha_2.$$

The rest of the proof is similar to the proof of Theorem 3.

Q.E.D.

Proof of Theorem 4:

Let $u_{jk}^a = v_{jk}^a - v_{mk}^a - v_{jn}^a + v_{mn}^a,\ \ u_{kj}^b = v_{kj}^b - v_{km}^b - v_{nj}^b + v_{nm}^b$, and let the vectors $x \in R_+^{m-1}$ and $y \in R_+^{n-1}$ have components defined by

$$x_j = \sum\nolimits_{s=1}^{n-1} u_{js}^a P_s^b, \quad y_k = \sum\nolimits_{s=1}^{m-1} u_{ks}^b P_s^a$$

and define

$$F_p^b(y) = \frac{\exp y_p}{1+\sum\nolimits_{s=1}^{n-1}\exp y_s}, \quad F_q^a(x) = \frac{\exp x_q}{1+\sum\nolimits_{s=1}^{m-1}\exp x_s}\ ,$$

$$G_j^a(y) = \sum\nolimits_{s=1}^{n-1} u_{js}^a F_s^b(y) \quad \text{and} \quad G_k^b(x) = \sum\nolimits_{s=1}^{m-1} u_{ks}^b F_s^a(x).$$

Let $\tilde{x}$ and x be two different vectors in R_+^{m-1} and define the norm of x by $\|x\| = \max_k |x_k|$.

Then by the mean value theorem we have that

(A.3) $$|G_k^b(\tilde{x}) - G_k^b(x)| = |\sum_{s\le m}\frac{\partial G_k^b(x^*)}{\partial x_s}(\tilde{x}_s - x_s)|$$

$$\le \sum_{s\le m}|\frac{\partial G_k^b(x^*)}{\partial x_s}||\tilde{x}_s - x_s| \le \max_{s\le m}|\tilde{x}_s - x_s|\sum_{s\le m}|\frac{\partial G_k^b(x^*)}{\partial x_s}|$$

for some suitable vector x^* in R_+^m. From (A.3) we conclude that

(A.5) $$\|G^b(\tilde{x}) - G^b(x)\| \le \|\tilde{x} - x\| \sum_{s\le m}\max_{k\le n}\max_{z\in R^m}|\frac{\partial G_k^b(z)}{\partial x_s}|.$$

From the definitions above it follows that

(A.6) $$\frac{\partial F_s^a}{\partial x_s} = F_s^a(1-F_s^a) \quad \text{and} \quad \frac{\partial F_s^a}{\partial x_r} = -F_s^a F_r^a,$$

for $r \neq s$. Hence, it follows that

(A.7) $$\frac{\partial G_k^b}{\partial x_s} = u_{ks}^b F_s^a(1-F_s^a) - F_s^a \sum_{r\neq s} u_{kr}^b F_r^a.$$

Let $d_k = \min_j u_{kj}^b$. Evidently, we have that $u_{kr}^b - d_k \geq 0$ and $\sum_r F_r^a < 1$ implying that

(A.8) $$0 \leq F_s^a \sum_{r\neq s}(u_{kr}^b - d_k)F_r^a \leq F_r^a(1-F_r^a)\max_{r\neq s}(u_{sr}^b - d_k).$$

Consequently, (A.7) and (A.8) yield

(A.9) $$|\frac{\partial G_k^b}{\partial x_s}| = |F_s^a u_{ks}^b - \sum_r F_r^a F_s^a u_{kr}^b| = F_s^a \,|(u_{ks}^b - d_k - \sum_r F_r^a(u_{kr}^b - d_k)|$$

$$\leq \max(F_s^a(1-F_s^a)(u_{sk}-d_k), \sum_{r\neq s} F_r^a F_s^a(u_{kr}^b - d_k)) \leq F_s^a(1-F_s^a)(\max_r u_{kr}^b - d_k).$$

It is easily verified that

(A.10) $$\sum_s F_s^a(1-F_s^a) \leq 1 - \frac{1}{m}$$

Thus, (A.5), (A.9) and (A.10) imply that

(A.11) $$\max_k \sum_{s\leq m} \max_{z\in R^m} |\frac{\partial G_k^b(z)}{\partial x_s}| \leq \max_k(\max_r u_{kr}^b - d_k)\left(1-\frac{1}{m}\right).$$

Therefore, we obtain from (A.5) and (A.11) that

(A.12) $$\| G^b(\tilde{x}) - G^b(x) \| \leq \| \tilde{x} - x \| \max_k(\max_r u_{kr}^b - d_k)\left(1-\frac{1}{m}\right).$$

Similarly, with $f_r = \min_k u_{rk}^a$ it follows that

(A.13) $$\| G^a(\tilde{y}) - G^a(y) \| \leq \| \tilde{y} - y \| \max_r(\max_k u_{rk}^a - f_r)\left(1-\frac{1}{n}\right).$$

Let

$$c = \max\left(\max_k(\max_r u_{rk}^a - d_k)\left(1-\frac{1}{m}\right), \max_r(\max_k u_{rk}^a - f_r)\left(1-\frac{1}{n}\right)\right).$$

From (A.12) and (A.13) it follows that

(A.14) $$\| G^b(\tilde{x}) - G^b(x) \| \leq c \| \tilde{x} - x \|$$

and

(A.15) $$\| G^a(\tilde{y}) - G^a(y) \| \leq c \| \tilde{y} - y \|.$$

Thus, (A.14) and (A.15) imply that if $c<1$ then the mapping $(G^a(y), G^b(x))$ is a contraction mapping on R_+^{m+n-2}. From the contraction mapping theorem (Rudin, 1976) it follows that a unique fixed point of $(G^a(x), G^b(y))$ exists.

Q.E.D.

Proof of Corollary 1:

It follows from (3.4) that

$$\theta_{rs} \leq \frac{\exp(\alpha_r)+\exp(\alpha_s)}{\exp(\alpha_r)+\exp(\alpha_s)+\exp(-\beta)\sum_{k\neq r,s}\exp(\alpha_k)} =: \tilde{\theta}_{rs}.$$

From Theorem 1(i) we obtain that if $\beta\theta \leq \beta\tilde{\theta}_{rs} < 2$ for all r and s then only one equilibrium exists.

Q.E.D.