Equilibria in Logit Models of Social Interaction and Quantal Response Equilibrium

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Abstract

The Quantal Response Equilibrium (QRE) extends the notion of Nash equilibrium in game theory to a corresponding stochastic equilibrium model. In QRE models, perfectly rational expectations equilibrium embodied in mixed strategy Nash equilibrium is replaced by an imperfect, or noisy, rational expectations equilibrium. An important subclass of QRE is the logit models of social interaction. It is known that at least one equilibrium exists in QRE models, but it is not known if, and when, there exist several equilibria. In this paper we discuss cases when unique- or several equilibria exist in two-persons multinomial logit QRE models. Second, we consider the equilibria in multinomial models with social interaction. Third, we discuss corresponding dynamic games and stability. Finally, we consider several examples.

Key words: Stochastic game theory, Logit QRE, Logit models with social interaction, Multiple equilibria

JEL classification: C02; C25; C62; C72; C73

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1. Introduction

Models with social interaction represent attempts to take into account that behavior of an individual in some contexts depends on the behavior of others. Thus, this type of models allows the researcher to address how individual and aspects of social behavior interact, consistent with typical views in social science (Coleman, 1988, 1990). Models with social interactions have been applied to a wide variety of problems within economics as well as within social science, see for example Durlauf (1997), Kirman (1997), and Rosser (1999) for overviews in social science. Several authors, such as Becker (1974, 1991), Becker and Murphy (2000), Brock and Durlauf (2001a, 2001b, 2002, 2006), Kirman and Zimmermann (2001), Manski (2000) and Shelling (1971), have discussed different model settings with social interaction in economics.

Models with social interaction can be viewed as special cases of the so-called Quantal Response Equilibrium models (QRE). Whereas the standard textbook approach to game theory assumes that players behave perfectly rational and are able to account for other players' uncertain actions in a consistent (optimal) way when computing and maximizing (expected) payoffs. McKelvey and Palfrey (1995, 1998) extended the notion of Nash equilibrium in game theory by allowing for randomness in behavior. The resulting theory is denoted Quantal Response Equilibrium (QRE). In the QRE model, perfectly rational expectations equilibrium embodied in mixed strategy Nash equilibrium is replaced by an imperfect, or noisy, rational expectations equilibrium meaning that the players are assumed to maximize expected utility plus noise (Goeree et al. 2005, 2016). The QRE comprises a limiting case where the QRE coincides with a subset of Nash equilibria (Nash, 1950). Related approaches are discussed by Anderson et al. (2002) and Chen et al. (1997). Haile et al. (2008) have discussed the empirical content of QRE and Melo et al. (2019) have discussed testing of QRE models.

It is known that QRE models have at least one equilibrium (McKelvey and Palfrey, 1995). However, despite its relevance, little is known about the uniqueness or number of equilibria in QRE models. To know the number of equilibria is useful knowledge when searching for equilibria. This issue is not only of theoretical interest because it has implications in the design of experiments, testing, and estimation of models involving this equilibrium concept (Aradillas-Lopez, 2020, Paula, 2017, and Melo et al., 2019). The number and location of equilibria also have implications for policy making in matters of conflict and collaboration.

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The purpose of this paper is to establish conditions for the existence of single versus multiple QRE in the two-persons multinomial logit QRE model. Second, we discuss criteria for stability of equilibria. Finally, we show how the results obtained for QRE can be used to characterize the set of equilibria in specific multinomial logit models with social interaction where the preferences of an individual depend on the aggregate behavior of others (Brock and Durlauf, 2001a, 2001b, 2002, 2003, Kirman and Zimmermann, 2001, and Kline and Tamer, 2020).

Melo (2022) has established uniqueness of a QRE for a broad class of n-person games. He shows that the uniqueness of a QRE is determined by a precise relationship between a measure of players' payoff concavity, a bound on the intensity of strategic interaction, and the number of players in the game. However, his results do not cover the cases treated in our paper. In the present paper, no assumption is made about strategic network interaction beyond the information represented by the payoff matrix. In this setting several equilibria are possible. In several applications the existence of multiple equilibria seems plausible and can be given intuitive interpretations, for example in settings where tipping points may occur (Harré and Bossomaier, 2014). By proposing a simple binary model of social interaction where several equilibria are possible. Becker (1991) provided a convincing explanation of behavior and price-setting of restaurants, plays, and sporting events. Specifically, he shows how one can explain why restaurants and other activities do not raise prices even with persistent excess demand.

The paper is organized as follows. In section 2 we present a general framework for the two-person multinomial logit QRE games. In section 3 we obtain criteria for the existence of multiple and unique equilibria in two-person binary QRE models and the results are generalized to the multinomial case in section 4. In section 5 we study equilibria in multinomial logit models with social interaction. Section 6 discusses the issue of stability and also shows that under specific conditions QRE is the unique solution of a corresponding contraction mapping and in section 7 we discuss special cases.

2. Multinomial logit quantal response games

Consider a setting with two players, player *a* and player *b*. There are *m* alternatives available to player *a* and *n* alternatives available to player *b*. Given that player *b* chooses alternative *k* then if player *a* chooses alternative *j* player *a* receives payoff v_{jk}^{a} . Similarly, the payoff to player *b* is v_{jk}^{b} if player *b* chooses alternative *k* given that player *a* chooses alternative *j*. The

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payoff matrices $\{v_{jk}^a\}$ and $\{v_{jk}^b\}$ are assumed to be common knowledge. The players are assumed to choose the strategy that maximizes expected payoff plus noise, where the noise is represented by a random variable. Consequently, the index of the chosen alternative becomes stochastic. Let P(j) be the probability that player *a* chooses alternative *j* and Q(k) the probability that player *b* chooses alternative *k* and let

$$P=(P(1),P(2),...,P(m-1)), \ Q=(Q(1),Q(2),...,Q(n-1)).$$

The corresponding combination of expected payoffs and noise for players *a* and *b* are given by

(2.1)
$$U_{j}^{a}(Q) = \sum_{r=1}^{n} v_{jr}^{a}Q(r) + \varepsilon_{j}^{a} \text{ and } U_{k}^{b}(P) = \sum_{r=1}^{m} v_{kr}^{b}P(r) + \varepsilon_{k}^{b}$$

where ε_j^a and ε_k^b are random variables. In equilibrium (Quantal Response Equilibrium, QRE) we have

$$P(j) = P(U_i^a(Q) = \max_r U_r^a(Q))$$
 and $Q(k) = P(U_k^b(P) = \max_r U_r^b(P))$

where the utilities of players *a* and *b* are given in (2.1) as functions of $\{P(j)\}\$ and $\{Q(k)\}\$, respectively. We make the following assumption.

Assumption 1

The random error terms $\{\varepsilon_j^a, \varepsilon_k^b\}$ are i.i.d. and independent of $\{P(j), Q(k)\}$ with *Gumbel c. d. f., i.e.*

$$P(\varepsilon_j^a \le x) = \exp(-\exp(-x/\lambda^a))$$
 and $P(\varepsilon_k^b \le x) = \exp(-\exp(-x/\lambda^b))$

for real x, where $\lambda^a > 0$ and $\lambda^b > 0$ are constants.

Assumption 1 means that the distributions of $\{\varepsilon_j^a, \varepsilon_k^b\}$ do not depend on the payoff matrices. Remember that the property that the random error terms are generated by Gumbel c.d.f. can be rationalized by the Independence of Irrelevant Alternatives assumption (IIA) proposed by Luce (1959), McFadden (1974) and Yellott (1977). The IIA assumption can be viewed as a representation of probabilistic rationality, see (Luce, 1977).

It follows from Assumption 1 that the logit QRE is given by

(2.2)
$$P(j) = P\left(U_{j}^{a} = \max_{s \le m} U_{s}^{a}\right) = \frac{\exp(\sum_{r=1}^{n} \lambda^{a} v_{jr}^{a} Q(r))}{\sum_{s=1}^{m} \exp(\sum_{r=1}^{n} \lambda^{a} v_{sr}^{a} Q(r))}$$

and

(2.3)
$$Q(k) = P\left(U_k^b = \max_{s} U_s^b\right) = \frac{\exp(\sum_{r=1}^m \lambda^b v_{rk}^b P(r))}{\sum_{s=1}^m \exp(\sum_{r=1}^m \lambda^b v_{rs}^b P(r))}$$

(McFadden, 1974). Let $F_j^a(Q)$ and $F_k^b(P)$ denote the multinomial logit expressions on the right hand side of (2.2) and (2.3), respectively, where Q(n) is replaced by $1 - \sum_{r \le n-1} Q(r)$ and P(m) is replaced by $1 - \sum_{r \le m-1} P(r)$. The equations above can thus be expressed more compactly as $P = F^a(Q)$ and $Q = F^b(P)$, or (P,Q) = F(P,Q) where $F(P,Q) = (F^a(Q), F^b(P)), F^a(Q) = (F_1^a(Q), F_2^a(Q), ..., F_{m-1}^a(Q))$ and $F^b(P) = (F_1^b(P), F_2^b(P), ..., F_{n-1}^b(P))$. Let $u_{jk}^a = \lambda^a (v_{jk}^a - v_{mk}^a - v_{jn}^a + v_{mn}^a)$ and $u_{jk}^b = \lambda^b (v_{jk}^b - v_{mk}^b - v_{jn}^b + v_{mn}^b)$. With no essential loss of generality we shall in the following

normalize by letting $\lambda^a = \lambda^b = 1$. Hence, the model in (2.2) and (2.3) can also be expressed as

(2.4)
$$P(j) = \frac{\exp(\sum_{r=1}^{n-1} u_{jr}^{a} Q(r) + v_{jn}^{a} - v_{mn}^{a})}{1 + \sum_{s=1}^{m-1} \exp(\sum_{r=1}^{n-1} u_{sr}^{a} Q(r) + v_{sn}^{a} - v_{mn}^{a})}$$

and

(2.5)
$$Q(k) = \frac{\exp(\sum_{r=1}^{m-1} u_{rk}^{b} P(r) + v_{mk}^{b} - v_{mn}^{b})}{1 + \sum_{s=1}^{n-1} \exp(\sum_{r=1}^{m-1} u_{rs}^{b} P(r) + v_{ms}^{b} - v_{mn}^{b})}$$

When m = n, $\lambda^a = \lambda^b = \lambda$ and $v_{jk}^a = v_{jk}^b =: v_{jk} = v_{kj}$ the model reduces to a symmetric logit QRE with Q(j) = P(j) where

(2.6)
$$P(j) = \frac{\exp(\sum_{r=1}^{m-1} u_{jr} P(r) + v_{jm} - v_{mm})}{1 + \sum_{s=1}^{m-1} \exp(\sum_{r=1}^{m-1} u_{sr} P(r) + v_{sm} - v_{mm})}$$

McKelvey and Palfrey (1995, p. 12) have proved that there exists at least one equilibrium in the general QRE model. For the sake of completeness we state the existence of an equilibrium in the case of the logit QRE. To this end let Δ_n be the simplex defined by

$$\Delta_p = \{ x \in \mathbb{R}^p : x_j \ge 0, \sum_{r=1}^p x_r \le 1 \}.$$

and let $F(x, y) = (F^{a}(y), F^{b}(x))$ be defined on $\Delta_{m-1} \times \Delta_{n-1}$. Clearly, F maps $\Delta_{m-1} \times \Delta_{n-1}$ into $\Delta_{m-1} \times \Delta_{n-1}$.

Theorem 1

The mapping F(P,Q) has a fixed point in $\Delta_{m-1} \times \Delta_{n-1}$.

The proof of Theorem 1 follows from Brouwer's fixed point theorem since $\Delta_{m-1} \times \Delta_{n-1}$ is compact and convex.

The stochastic formulation of game theory enables researchers to formulate and estimate models in cases where the payoffs are unobservable utilities that may depend on several observable attributes, pecuniary as well as non-pecuniary ones. For example, in the symmetric case where $\lambda v_{jk} = Z_{jk}\beta$, $Z_{jk} = (Z_{jk}(1), Z_{jk}(2), ...)$, is a vector of observable attributes and β is an unknown parameter vector to be estimated, the QRE model implies that

$$\log\left(\frac{P(j)}{P(m)}\right) = \sum_{k} P(k)(Z_{jk} - Z_{mk})\beta.$$

We note that when the dimension of β is less than or equal to m-1 then β is identified provided the matrix $\{a_{ir}\}$ has rank m-1, where

$$a_{jr} = \sum_{k} P(k)(Z_{jk}(r) - Z_{mk}(r)).$$

To simplify notation, we shall in the following normalize such that $\lambda^a = \lambda^b = 1$. This, simply means that λ^a and λ^b are absorbed in the respective payoff matrices and thus this normalization represents no loss of generality.

Proposition 1

The equilibrium choice probabilities given in (2.2) and (2.3) satisfy the following inequalities:

$$\frac{\exp(\min_{s} v_{js}^{a})}{\exp(\min_{s} v_{js}^{a}) + \sum_{r \neq j} \exp(\max_{s} v_{rs}^{a})} \le P(j) \le \frac{\exp(\max_{s} v_{js}^{a})}{\exp(\max_{s} v_{js}^{a}) + \sum_{r \neq j} \exp(\min_{s} v_{rs}^{a})}$$

and

$$\frac{\exp(\min_{s} v_{sk}^{b})}{\exp(\min_{s} v_{sk}^{b}) + \sum_{r \neq k} \exp(\max_{s} v_{sr}^{b})} \leq Q(k) \leq \frac{\exp(\max_{s} v_{sk}^{b})}{\exp(\max_{s} v_{sk}) + \sum_{r \neq k} \exp(\min_{s} v_{sr}^{b})}.$$

The proof of Proposition 1 is given in the appendix. Proposition 1 determines the range of possible equilibria.

3. Equilibria in the binary logit QRE model

In this section we shall analyze the binary case where m = n = 2. Let P = P(1), Q = Q(1),

$$u^{a} = v_{11}^{a} + v_{22}^{a} - v_{12}^{a} - v_{21}^{a}, \ u^{b} = v_{11}^{b} + v_{22}^{b} - v_{12}^{b} - v_{21}^{b},$$
$$L(x) = \frac{1}{1 + e^{-x}}$$

and

(3.1)
$$g(x) = \log\left(\frac{x}{1-x}\right) - u^a L(u^b x + v_{21}^b - v_{22}^b).$$

For $x \in (0,1)$. In this case the QRE model reduces to

(3.2)
$$P = L(u^a Q + v_{12}^a - v_{22}^a)$$

and

(3.3)
$$Q = L(u^{b}P + v_{21}^{b} - v_{22}^{b}).$$

When (3.3) is inserted into (3.2) we get

(3.4)
$$P = L(u^{a}L(u^{b}P + v_{12}^{b} - v_{22}^{b}) + v_{12}^{a} - v_{22}^{a})$$

which implies that

(3.5)
$$g(P) = v_{12}^a - v_{22}^a$$
.

Since g is continuous, $g(0) = -\infty$ and $g(1) = \infty$ it follows from Bolzano's theorem that there exists at least one solution of (3.5) for P. However, (3.5) may have several solutions for P, depending on the payoff matrices, $\{v_{jk}^a\}$ and $\{v_{jk}^b\}$. Before we state the next result we need $-\infty$ the following lemma.

Lemma 1

Let

$$C_1 = \max_{x \in [0,1]} \{ x(1-x)L(u^b x + v_{21}^b - v_{22}^b)(1 - L(u^b x + v_{21}^b - v_{22}^b)) \}.$$

When $u_a u_b C_1 > 1$ the function

$$u^{a}u^{b}x(1-x)L(u^{b}x+v_{21}^{b}-v_{22}^{b})(1-L(u^{b}x+v_{21}^{b}-v_{22}^{b}))-1$$

as two real roots.

The proof of Lemma 1 is given in the Appendix. The next Theorem gives a complete account of the equilibria in the binary QRE model.

Theorem 2

Assume that Assumption 1 holds and that m = n = 2. When $u_a u_b C_1 > 1$, let w_1 and w_2 , $w_1 \le w_2$, be the roots of the function

$$(3.6) u^a u^b x(1-x) L(u^b x + v_{21}^b - v_{22}^b)(1 - L(u^b x + v_{21}^b - v_{22}^b)) - 1.$$

If either of the 3 conditions

(i)
$$u^{a}u^{b}C_{1} < 1$$
,
(ii) $u^{a}u^{b}C_{1} > 1$ and $\log\left(\frac{w_{1}}{1-w_{1}}\right) - u^{a}L(u^{b}w_{1}+v_{21}^{b}-v_{22}^{b}) < v_{12}^{a}-v_{22}^{a}$,

or

(*iii*)
$$u^{a}u^{b}C_{1} > 1$$
 and $\log\left(\frac{w_{2}}{1-w_{2}}\right) - u^{a}L(u^{b}w_{2} + v_{21}^{b} - v_{22}^{b}) > v_{12}^{a} - v_{22}^{a}$

hold there exists a unique set of equilibrium probabilities of the binary logit QRE model. If instead

(iv)
$$u^{a}u^{b}C_{1} > 1$$
 and
 $\log\left(\frac{w_{2}}{1-w_{2}}\right) - u^{a}L(u^{b}w_{2} + v_{21}^{b} - v_{22}^{b}) < v_{12}^{a} - v_{22}^{a} < \log\left(\frac{w_{1}}{1-w_{1}}\right) - u^{a}L(u^{b}w_{1} + v_{21}^{b} - v_{22}^{b})$

there exist 3 equilibrium probabilities determined by (3.3).

The proof of Theorem 2 is given in the appendix.

In the symmetric binary case where $v_{jk}^a = v_{jk}^b = v_{jk} = v_{kj}$, we have, with P = P(1), that

(3.7)
$$P = L(uP + v_{12} - v_{22})$$

which is equivalent to

(3.8)
$$\psi(P) =: \log\left(\frac{P}{1-P}\right) - uP = v_{12} - v_{22}.$$

Since

$$\psi'(x) = \frac{1}{x(1-x)} - u$$





we realize that unless u > 4 no more than one equilibrium can occur because the highest value x(1-x) can attain is 0.25. Figure 1 illustrates the nature of the equilibria in the symmetric binary logit QRE model in the case with three equilibria. Thus, the plot in Figure 1 only has this form when u > 4. When $\alpha_L < \psi(P) < \alpha_H$ three equilibria are possible, namely P^r , r = A, B, C given in Figure 1.

4. Equilibria in the multinomial case

We now turn to an analysis of the multinomial logit QRE model.

Theorem 3

Consider the following inequalities

(i)
$$\sum_{r} [\max_{s} (v_{rs}^{a} - v_{ms}^{a}) - \min_{s} (v_{rs}^{a} - v_{ms}^{a})] < 4,$$

(*ii*)
$$\sum_{r} [\max_{s} (v_{rs}^{b} - v_{ms}^{b}) - \min_{s} (v_{rs}^{b} - v_{ms}^{b})] < 4,$$

(iii)
$$\max_{r} (\max_{s} (v_{sr}^{a} - v_{sn}^{a}) - \min_{s} (v_{sr}^{a} - v_{sn}^{a})) < \frac{m-1}{m-2}$$

and

(*iv*)
$$\max_{r} (\max_{s} (v_{rs}^{b} - v_{ms}^{b}) - \min_{s} (v_{rs}^{b} - v_{ms}^{b})) < \frac{n-1}{n-2}.$$

The mapping F(P,Q) is a contraction if either (i) and (ii), or (iii) and (iv) are satisfied and F therefore has a unique fixed point on $\Delta_{m-1} \times \Delta_{n-1}$.

The proof of Theorem 3 is given in the Appendix. Theorem 3 states that when the inequalities given in Theorem 3 are fulfilled there exists a unique equilibrium. However, when these inequalities do not hold several equilibria may occur.

Let \tilde{P} and \tilde{Q} be any equilibrium vectors and let $\theta_a = \tilde{P}(1) + \tilde{P}(2)$ and $\theta_b = \tilde{Q}(1) + \tilde{Q}(2)$. Moreover, let $p = P(1) / \theta_a$ and $q = Q(1) / \theta_b$. For $p, q \in (0,1)$ it follows from (2.4) and (2.5) that

(4.1)
$$p = L(\theta_b u^a q + K_a)$$

and

(4.2)
$$q = L(\theta_a u^b p + K_b)$$

where

$$K_{a} = v_{1n}^{a} - v_{2n}^{a} + (u_{12}^{a} - u_{22}^{a})\theta_{b} + \sum_{r=3}^{n-1} (u_{1r}^{a} - u_{2r}^{a})\tilde{Q}(r)$$

and

$$K_{b} = v_{m1}^{b} - v_{m2}^{b} + (u_{21}^{b} - u_{22}^{b})\theta_{a} + \sum_{r=3}^{m-1} (u_{r1}^{b} - u_{r2}^{b})\tilde{P}(r).$$

If (4.2) is inserted into (4.1) we get

(4.3)
$$p = L(\theta_b u^a L(\theta_a u^b p + K_b) + K_a).$$

Thus, as in the binary case the problem of solving for p and q in (4.1) and (4.2) can be reduced to a one-dimensional problem of finding the solutions to (4.3), which is similar to (3.4). From Theorem 2 we therefore get the next result.

Theorem 4

Assume that Assumption 1 holds. Let \tilde{P} and \tilde{Q} be any equilibrium vectors and let $\tilde{P}(1) + \tilde{P}(2) =: \theta_a$ and $\tilde{Q}(1) + \tilde{Q}(2) =: \theta_b$. Let

$$C_2 = \max_{x \in [0,1]} \{ x(1-x)L(\theta_a u^b x + K_b)(1 - L(\theta_a u^b x + K_b)) \}$$

and let ω_1 and ω_2 , $\omega_1 \leq \omega_2$, be the roots of the function

$$\theta_a \theta_b u^a u^b (1-x) x L(\theta_a u^b x + K_b) (1 - L(\theta_a u^b x + K_b)) - 1$$

when $\theta_a \theta_b u^a u^b C_2 > 1$.

If

$$\theta_a \theta_b u^a u^b C_2 > 1$$

and

$$\log\left(\frac{\omega_2}{1-\omega_2}\right) - \theta_b u^a L(\theta_a u^b \omega_2 + K_b) < K_a < \log\left(\frac{\omega_1}{1-\omega_1}\right) - \theta_b u^a L(\theta_a u^b \omega_1 + K_b)$$

there exist three equilibria determined by (4.1) and (4.2).

Theorem 4 means that for any equilibrium vectors \tilde{P} and \tilde{Q} then, keeping $\{\tilde{P}(r), r \ge 3\}$ and $\{\tilde{Q}(r), r \ge 3\}$ fixed, there may exist several equilibrium values of P(1) and Q(1). Note that the enumeration of the alternatives is irrelevant for the result of Theorem 4. If for example m = 3, θ_a may be $\tilde{P}(1) + \tilde{P}(2)$, $\tilde{P}(1) + \tilde{P}(3)$ or $\tilde{P}(2) + \tilde{P}(3)$. In the case of multiple equilibria it is easy to compute the equilibrium probabilities when one equilibrium has been found. When $\theta_a = \tilde{P}(1) + \tilde{P}(2)$ and p have been determined from (4.3) then the corresponding equilibrium probabilities are $P(1) = p\theta_a$ and $P(2) = \theta_a - \theta_a p$.

Next, we shall consider the symmetric case where $v_{jk}^a = v_{jk}^b = v_{jk} = v_{kj}$. In this case the equations in (4.1) and (4.2) reduce to a single equation, namely

(4.5)
$$\log\left(\frac{p}{1-p}\right) - \theta u p = v_{1m} - v_{2m} + (u_{12} - u_{22})\theta + \sum_{r=3}^{m-1} (v_{1r} - v_{2r})\tilde{P}(r)$$

where $u = v_{11} + v_{22} - v_{12} - v_{21}$ and $\theta = \tilde{P}(1) + \tilde{P}(2)$. Let

(4.6)
$$f(x) = 0.5\sqrt{x(x-4)} - \log(0.5x - 1 + 0.5\sqrt{x(x-4)})$$

for $x \ge 4$. It is easy to verify that f(x) is strictly increasing for x > 4.

Corollary 1

Assume that Assumption 1 holds and that $v_{jk}^a = v_{jk}^b = v_{jk} = v_{kj}$. Let \tilde{P} be any equilibrium vector of choice probabilities and let $\theta = \tilde{P}(1) + \tilde{P}(2)$.

If $u\theta > 4$ and $f(u\theta) > |v_{1m} - v_{2m} + (u_{12} - u_{22})\theta + \sum_{r=3}^{m} (u_{1r} - u_{2r})\tilde{P}(r)|$ there exist three equilibria determined by (4.5).

The proof of Corollary 1 is given in the appendix.

Next, we shall consider the binary zero sum QRE model. In this case we have the following result which is an immediate implication of Theorem 3.

Corollary 2

Assume that Assumption 1 holds, and that $v_{jk}^b = -v_{jk}^a$. Consider the inequalities

(i)
$$\sum_{r} [\max_{s} |v_{rs}^{a} - v_{ms}^{a}| - \min_{s} |v_{rs}^{a} - v_{ms}^{a}|] < 4$$

and

(*ii*)
$$\max_{s} (\max_{s} |v_{sr}^{a} - v_{sn}^{a}| - \min_{s} |v_{sr}^{a} - v_{sn}^{a}|) < \min\left(\frac{m-1}{m-2}, \frac{n-1}{n-2}\right)$$

The mapping F(P,Q) is a contraction if either (i) or (ii) are satisfied and F therefore has a unique fixed point on $\Delta_{m-1} \times \Delta_{n-1}$.

5. The multinomial logit model with social interaction

Recall that by social interaction, it is understood interdependences between the decisions made by individuals which are not mediated by markets. Many interactions-based models are variants of game-theoretic models (Blume, 1997, Young, 1998, and Morris, 2000). What distinguishes the research on interactions-based models is the explicit attention given to formulating how an individual's behavior is a function of the behavior of others and then studying what aggregate properties emerge in the population.

In a typical model with social interaction in the case of observational identical individuals the utility function of individual *i* is given by

(5.1)
$$U_{ij} = \alpha_j + \beta P(j) + \varepsilon_{ij}$$

where α_j is a deterministic term that may depend on observable attributes of alternative *j* and P(j) is the probability that a decision-maker shall choose alternative *j*. The intuition of the preference structure given in (5.1) is that in many situations, such as the choice among restaurants (Becker, 1991), choice among books or cultural events, an individual's preferences may depend on the aggregate behavior of others. The preference structure given in (5.1) may be extended to allow $\{\alpha_j\}$, and possibly β , to depend on observable individual characteristic in addition to the alternative-specific attributes.

Under assumption 1, with λ normalized to 1 it follows that

(5.2)
$$P(j) = \frac{\exp(\alpha_j + \beta P(j))}{\sum_{r=1}^{m} \exp(\alpha_r + \beta P(r))}$$

$$=\frac{\exp(\alpha_{j}-\beta+2\beta P(j)+\beta\sum_{r\neq j}^{m-1}P(r))}{1+\sum_{r< m}\exp(\alpha_{r}-\beta+2\beta P(r)+\beta\sum_{s\neq r}^{m-1}P(s))}=:H(P)$$

with $\alpha_m = 0$. We note that the structure of the formula in (5.2) is a special case of the QRE framework considered above with

(5.3)
$$v_{jk} = \alpha_j + \beta \delta_{jk}$$

where $\delta_{jk} = 1$ if j = k, and zero otherwise. The next result follows readily from Theorem 3 because (5.3) implies that

$$\sum_{r < m} [\max(0, \max_{j < m} u_{jr}) - \min(0, \min_{j < m} u_{jr})] = 2\beta$$

Corollary 3

Assume that (5.3) holds. If $|\beta| < 2$ the mapping H given in (5.2) is a contraction and there exists a unique equilibrium vector of choice probabilities.

The next corollary follows from Corollary 1.

Corollary 4

Let \tilde{P} be any equilibrium vector of choice probabilities satisfying (5.2) and let $\theta = \tilde{P}(1) + \tilde{P}(2).$

If $\beta\theta > 2$ and $f(2\beta\theta) > |\alpha_1 - \alpha_2|$

there exist at least three equilibria.

Evidently, as noted above, the enumeration of the alternatives is irrelevant for the result of the theorem. Thus, if $\theta = \tilde{P}(r) + \tilde{P}(s)$ where $r \neq s$ then the result of Corollary 4 holds with the obvious modification of the indexation of the alfas.

Corollary 5

Assume that $\alpha_j = 0$ for all *j* in the multinomial logit model with social interaction. If $\beta < 2$ there exist a unique equilibrium with choice probabilities equal to 1/m. If $\beta > m$ there exist at least three equilibria which are determined by $P_j(1) = p_j / m$, $P_j(2) = (1 - p_j) / m$, $P_j(r) = 1/m$ for $r \ge 3$, j = 1, 2, 3, where $p_1 = 0.5$, p_2 and p_3 are the other solutions to

$$\log\left(\frac{p}{1-p}\right) + 2\beta(0.5-p)/m = 0.$$

The proof of Corollary 5 is given in the appendix. Corollary 5 extends Theorem 1 in Brock and Durlauf (2006). In the binary case that corresponds to the setting discussed by Becker (1991) the model in (5.2), with the normalization $\alpha_2 = 0$, reduces to

(5.4)
$$P(1) = \frac{1}{1 + \exp(-\alpha_1 + \beta - 2\beta P(1))}$$

In the binary case the next result follows immediately from Corollary 4.

Corollary 6

Assume that m = 2 in the logit model with social interaction.

(*i*) If either $\beta < 2$

or

 $\beta > 2$ and $f(2\beta) < |\alpha_1 - \alpha_2|$

then there exists one equilibrium probability.

(ii) If $\beta > 2$ and $f(2\beta) > |\alpha_1 - \alpha_2|$

there exist three equilibrium probabilities.

The result of Corollary 6 has also been proved by Becker (1991) and Brock and Durlauf (2001).

6. Dynamic games and stability

In this section we consider settings with two type of players that play repeated games at discrete time epochs. The implied dynamic system is governed by the recursive relations

(6.1)
$$(P_{t+1}, Q_{t+1}) = F_t(P_t, Q_t)$$

t = 1, 2,..., where t indexes the time periods, and where it is understood that the payoffs may be time dependent. An important question is whether the non-linear difference equations in (6.1) converges towards a unique QRE when $F_t = F$. This feature corresponds to situations where the payoffs are time invariant during "long" time intervals. Let us first consider the binary case where m = n = 2. We have the following result.

Theorem 5

Assume that Assumption 1 holds and that m = n = 2. When $u^a u^b C_1 > 1$, let w_1 and w_2 , $w_1 \le w_2$ be the solutions to (3.6). If either

 $(i) \qquad u^a u^b C_1 < 1,$

(*ii*)
$$u^{a}u^{b}C_{1} > 1$$
 and $\log\left(\frac{w_{1}}{1-w_{1}}\right) - u^{a}L(u^{b}w_{1}+v_{12}^{b}-v_{22}^{b}) < v_{12}^{a}-v_{22}^{a}$

or

(iii)
$$u^a u^b C_1 > 1$$
 and $\log\left(\frac{w_2}{1 - w_2}\right) - u^a L(u^b w_2 + v_{12}^b - v_{22}^b) > v_{12}^a - v_{22}^a$

hold, the unique set of equilibrium probabilities is stable. If instead

(iv)
$$u^{a}u^{b}C_{1} > 1$$
 and
 $\log\left(\frac{w_{2}}{1-w_{2}}\right) - u^{a}L(u^{b}w_{2} + v_{12}^{b} - v_{22}^{b}) < v_{12}^{a} - v_{22}^{a} < \log\left(\frac{w_{1}}{1-w_{1}}\right) - u^{a}L(u^{b}w_{1} + v_{12}^{b} - v_{22}^{b})$

there exist one unstable and two stable sets of equilibrium probabilities.

The proof of Theorem 5 is given in the appendix.

Consider the symmetric binary case where the choice probability $P_t = P_t(1)$ is determined by

$$P_{t+1} = L(uP_t + v_{12} - v_{22}).$$

Corollary 7

In the binary logit model with social interaction there exists a unique stable equilibrium probability if either

(*i*)
$$\beta < 2$$
, or
(*ii*) $\beta > 2$ and $f(2\beta) < |\alpha_1 - \alpha_2|$.

If

(iii) If $\beta > 2$ and $f(2\beta) > |\alpha_1 - \alpha_2|$

there exist three equilibrium probabilities where two are stable and one is unstable.

The proof of Corollary 7 is given in the appendix. Becker (1991) has also given a proof of the result in Corollary 7.

Next, consider the general case. It follows from (2.4) and (2.5) that

$$\frac{\partial F_j^a(Q)}{\partial Q(d)} = P(j)(u_{jd}^a - \sum_{r \le m-1} P(r)u_{rd}^a), \quad \frac{\partial F_k^b(Q)}{\partial P(d)} = Q(k)(u_{dk}^b - \sum_{r \le n-1} Q(r)u_{dr}^b)$$

and

$$\frac{\partial F_j^a(Q)}{\partial P(d)} = \frac{\partial F_k^b(P)}{\partial Q(d)} = 0$$

for all *d*. Let J(P,O) be the Jacobian matrix of F(P,Q) and let *A* and *B* be the matrices with elements

$$a_{jd} = P(j)(u_{jd}^b - \sum_{r \le m-1} P(r)u_{rd}^b)$$
 and $b_{kd} = Q(k)(u_{dk}^a - \sum_{r \le n-1} Q(r)u_{dr}^a).$

Hence, we can express the Jacobian matrix as

$$J(P,Q) = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

Theorem 6

A fixed point (\tilde{P}, \tilde{Q}) (say) of the mapping F is stable if the absolute value of all the eigenvalues of $J(\tilde{P}, \tilde{Q})$ are less than one.

The result of Theorem 6 is well known; for a proof see for example Michel et al. (2008).

Corollary 8

Under the assumptions of Theorem 3 F has a unique and stable fixed point.

Under the conditions of Theorem 3 there exists an iteration process, starting with any initial vector of choice probabilities that are not degenerate will converge to a unique equilibrium.

The next result follows from Corollary 3.

Corollary 9

The mapping $P \rightarrow H(P)$ given in (5.2) has a unique stable equilibrium when $|\beta| < 2$.

7. Examples

Example 7.1

In this example we consider a two-person game with

(7.1) $\sum_{r} v_{jr}^{a} = \sum_{r} v_{1r}^{a}$ and $\sum_{r} v_{kr}^{b} = \sum_{r} v_{1r}^{b}$

for all *j* and *k*. Note that the restrictions in (7.1) do not rule out zero sum games. It follows immediately from (7.1) that $\tilde{P}(j) = 1/m$ and $\tilde{Q}(k) = 1/n$ are equilibrium probabilities satisfying (2.2) and (2.3). The corresponding theta values are therefore $\theta_a = 2/m$ and $\theta_b = 2/n$. In this case equations (4.1) and (4.2) become

(7.2)
$$p = L(2n^{-1}u^a(q-0.5))$$
 and $q = L(2m^{-1}u^b(p-0.5))$

which imply that *p* is determined by

(7.3)
$$p = L(2n^{-1}u^a(L(2m^{-1}u^b(p-0.5))-0.5))$$

which has the same structure as (4.3) with $K_a = -n^{-1}u^a$ and $K_b = -m^{-1}u^b$. In this case (4.4) becomes

(7.4)
$$4m^{-1}n^{-1}u^{a}u^{b}(1-x)xL(2m^{-1}(x-0.5))(1-L(2m^{-1}(x-0.5)))-1=0.$$

Furthermore,

$$C_{2} = \max_{x \in [0,1]} \{ x(1-x)L(2m^{-1}u^{b}(x-0.5))(1-L(2m^{-1}u^{b}(x-0.5))) \}.$$

It is easily verified that the function

$$x(1-x)L(2m^{-1}u^{b}(x-0.5))(1-L(2m^{-1}u^{b}(x-0.5)))$$

attains its unique maximum for x = 0.5, which implies that $C_2 = 1/16$. Furthermore, from Theorem 4 it follows that the roots ω_1 and ω_2 are determined as the solutions to (7.4). When $u^a u^b > 4mn$ and (7.1) holds we get from Theorem 4 that there are three solutions for p of (7.3) provided

$$\log\left(\frac{\omega_{2}}{1-\omega_{2}}\right) - \frac{2L(2m^{-1}u^{b}(\omega_{2}-0.5))}{n} < \frac{-u^{a}}{n} < \log\left(\frac{\omega_{1}}{1-\omega_{1}}\right) - \frac{2L(2m^{-1}u^{b}(\omega_{1}-0.5))}{n}.$$

The case with p = q = 0.5 corresponds to the equilibrium (\tilde{P}, \tilde{Q}) .

Example 7.2

In this example the game is assumed that (7.1) holds and that m = n and $v_{jk}^a = v_{jk}^b = v_{jk}$. Hence, in this case (7.1) can be expressed as

$$(7.5) \qquad \sum_{r} v_{jr} = \sum_{r} v_{1r}$$

for all *j*. The "battle of the sexes" game (Luce and Raiffa, 1957) has payoffs satisfying (7.5). It follows that the condition $u^a u^b > 4mn$ reduces to u > 2m. From Corollary 1 it follows that under (7.5) three equilibria exist provided u > 2m and $f(2m^{-1}u) > 0$. Since f(x) is strictly increasing for x > 4, it implies that the inequality $f(2m^{-1}u) > 0$ always holds in this case. In other words, when (7.5) holds there are always two equilibria when u > 2m in addition to the equilibrium (\tilde{P}, \tilde{Q}). The equilibria are determined by the roots of

$$\log\left(\frac{p}{1-p}\right) - \frac{2u(p-0.5)}{m}.$$

Example 7.3

Recall that an equilibrium can be stable even if the mapping H is not a contraction. Consider the threenomial model with social interaction. In this case (5.2) can be expressed as

$$P(1) = \frac{\exp(\alpha_1 - \beta + 2\beta P(1) + \beta P(2))}{1 + \exp(\alpha_1 - \beta + 2\beta P(1) + \beta P(2)) + \exp(\alpha_2 - \beta + \beta P(1) + 2\beta P(2))}$$

and

$$P(2) = \frac{\exp(\alpha_{1} - \beta + \beta P(1) + 2\beta P(1))}{1 + \exp(\alpha_{1} - \beta + 2\beta P(1) + \beta P(1)) + \exp(\alpha_{2} - \beta + \beta P(1) + 2\beta P(1))}$$

It follows readily that the Jacobian matrix J in this case becomes equal to

(7.6)
$$J = \begin{pmatrix} \beta(2P(3) + P(2))P(1) & \beta(P(3) - P(2))P(1) \\ \beta(P(3) - P(1))P(2) & \beta(2P(3) + P(1))P(2) \end{pmatrix}.$$

It follows that the larges eigenvalue that corresponds to the matrix J is equal to $\beta\mu$ where

$$\mu = P(1)P(2) + (1 - P(3))P(3) + \left[\left(P(1)P(2) + (1 - P(3))P(3) \right)^2 - 3P(1)P(2)P(3) \right]^{1/2}$$

From Theorem 6 it follows that an equilibrium vector P is stable if and only if $\beta \mu < 1$.

Example 7.4

This example is a special case of social interaction with m = 3 alternatives. Suppose $\beta = 3$, $\alpha_1 = \alpha_2 = \alpha = \log 2 - 1.6 \approx -0,907$. and $\alpha_3 = 0$. Then it is immediately verified that $\tilde{P}(1) = \tilde{P}(2) = 0.4$ and $\tilde{P}(3) = 0.2$ is an equilibrium solution to (5.2). Let $\theta = \tilde{P}(1) + \tilde{P}(2) = 0.8$. Since $\alpha_1 - \alpha_2 = 0$, $\beta \theta = 2.4 > 2$ and $f(2\beta\theta) > 0$, Corollary 4 applies implying that there are two additional equilibria determined by

$$\log\left(\frac{p}{1-p}\right) = 2\beta\theta p - \beta\theta = 4.8p - 2.4$$

The solutions to the above equation are $p_1 = 0.173$, $p_2 = 1 - p_1 = 0.827$ and $p_3 = \tilde{P}(1) / \theta = 0.5$. Hence, there is second equilibrium solution given by $P^*(1) = \theta p_1 = 0.138$, $P^*(2) = \theta - \theta p_1 = 0.662$ and $P^*(3) = 1 - \theta = 0.4$. The third equilibrium is given by $P^{**}(1) = P^*(2)$ and $P^{**}(2) = P^*(1)$.

Consider next the dynamic extension (Section 6) and let us find out which equilibria are stable. The formula for the Jacobian matrix associated with the model is given by (7.6). It follows that one of the eigenvalues are greater than 1 when $P = \tilde{P}$, which implies that this equilibrium is unstable. In contrast, both eigenvalues are positive and less than 1 when $P = P^*$. Thus, there are two stable equilibria where one of the stable equilibria is equivalent to the other.

Example 7.5

This example is also a special case of social interaction with m = 3, $\beta = 3$, $\alpha_1 = -0.146$, $\alpha_2 = -2.47$ and $\alpha_3 = 0$. Let $\tilde{P}(1) = 0.1$, $\tilde{P}(2) = 0.2$ and $\tilde{P}(3) = 0.7$, which is an equilibrium solution. With $\theta = \tilde{P}(1) + \tilde{P}(2) = 0.3$ we find that $\beta \theta = 0.9 < 2$ so that for this θ only one equilibrium

(7.7)
$$\log\left(\frac{p}{1-p}\right) = 2\beta\theta p - \beta\theta$$

exists. Consider next the case when $\theta = \tilde{P}(1) + \tilde{P}(3) = 0.8$. In this case $\beta\theta = 2.4 > 2$ and $f(2\beta\theta) = f(4.8) = 0.113 < |\alpha_1 - \alpha_3| = 0.146$ which means that also for this θ only one solution to (7.9) exists. Consider finally the case when $\theta = P(2) + P(3) = 0.9$. In this case $\beta\theta = 2.7 > 2$ and $f(2\beta\theta) = f(5.4) = 0.252 > |\alpha_2 - \alpha_3| = 0.247$ which implies that three solutions to (7.9) exists. With $p = P(2) / \theta = P(2) / 0.9$ it follows that $p_1 = 0.270$, $p_2 = 0.928$ and $p_3 = \tilde{P}(2) / \theta = 0.222$. Hence, the corresponding equilibrium probabilities become; $P^*(1) = 0.1$, $P^*(2) = 0.243$, $P^*(3) = 0.657$, $P^{**}(1) = 0.1$, $P^{**}(2) = 0.835$ and $P^{**}(3) = 0.065$, in addition to \tilde{P} .

Consider the dynamic extension (Section 6). By applying (7.6) we find that the equilibria and P^{**} is stable whereas \tilde{P} and P^{*} are unstable. The largest eigenvalue of the

Jacobian that corresponds to \tilde{P} is equal to 1.003 which means that this equilibrium is "close" to being stable.

8. Conclusion

In this paper we have established simple conditions which determine the number of stable equilibrium probabilities in the two-persons multinomial logit QRE models. Second, we have applied the obtained results to characterize the set of equilibrium probabilities in logit models of social interaction. Third, we have considered dynamic QRE games and discussed when they have stable equilibria. Finally, we have discussed some examples where multiple equilibria occur.

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Appendix

Proof of Proposition 1:

Note that the mapping

$$(y_1, y_2, ..., y_m) \rightarrow \frac{\exp(y_j)}{\sum_{r=1}^{m} \exp(y_r)}$$

from R^m to R is increasing in y_j and decreasing in y_k for $k \neq j$. Since $\{Q(r)\}$ add up to 1 it thus follows that

$$P(j) = \frac{\exp(\sum_{r} v_{jr}^{a} Q(r))}{\sum_{s} \exp(\sum_{r} v_{sr}^{a} Q(r))}$$
$$\leq \frac{\exp(\max_{r} v_{jr}^{a})}{\exp(\max_{r} v_{jr}^{a}) + \sum_{s \neq j} \exp(\min_{r} v_{sr}^{a})}.$$

Similarly, it follows that

$$P(j) \ge \frac{\exp(\min_{r} v_{jr}^{a})}{\exp(\min_{r} v_{jr}^{a}) + \sum_{s \neq j} \exp(\max_{r} v_{sr}^{a})}.$$

Hence, we have proved the first set of inequalities of the theorem. The proof of the second set is similar.

Proof of Lemma 1:

Let h(x) be defined by

(0,1). $h(x) = \log \{x(1-x)L(cx+d)(1-L(cx+d))\}.$

By differentiation we obtain that

$$h''(x) = -\frac{1}{x^2} - \frac{1}{(1-x)^2} - 2c^2L(cx+d)(1-L(cx+d)).$$

Since h''(x) < 0, h(x) is strictly concave. Evidently, h(x) tends towards $-\infty$ when x tends to zero or one. Thus, if a is a positive constant and $a \exp\left(\max_{x \in [0,1]} h(x)\right) > 1$ it follows that $a \exp(h(x)) - 1$ has two roots in (0,1).

Q.E.D.

Proof of Theorem 2:

Recall that

$$C_{1} = \max_{x \in [0,1]} \{ x(1-x)L(u^{b}x + v_{21}^{b} - v_{22}^{b})(1 - L(u^{b}x + v_{21}^{b} - v_{22}^{b})) \}$$

and

$$g(x) = \log\left(\frac{x}{1-x}\right) - \frac{u^a}{1 + \exp(-u^b x - v_{21}^b + v_{22}^b)}$$

for $x \in (0,1)$. Evidently, we get from (3.1) and (3.2) that in equilibrium

(A.1)
$$v_{12}^a - v_{22}^a = g(P).$$

By Lemma 1 it follows that it is concave and therefore has two roots, say w_1 and w_2 , $w_1 \le w_2$. We have that

(A.2)
$$g'(x) = \frac{1 - u^a u^b x (1 - x) L (u^b x + v_{12}^b - v_{22}^b) (1 - L (u^b x + v_{12}^b - v_{22}^b))}{x(1 - x)}$$

It follows from Lemma 1 that g'(x) has two roots, $w_1 \le w_2$, (say), when $u^a u^b C_1 > 1$.

Furthermore, g(x) increases until w_1 and thereafter decreases until w_2 , and then increases again. In contrast, if $u^a u^b C_1 < 1$, g(x) is an increasing function in (0,1). Therefore, if $u^a u^b C_1 < 1$, (A.1) has only one solution for *P*. If $u^a u^b C_1 > 1$ and $v_{12}^a - v_{22}^a = g(P) > g(w_1)$ or $v_{12}^a - v_{22}^a = g(P) < g(w_2)$ then also only one equilibrium can occur. If, however, $u^a u^b C_1 > 1$ and $g(w_2) < v_{12}^a - v_{22}^a < g(w_1)$, we realize that 3 equilibria exist.

Proof of Corollary 1:

Let $\theta = \tilde{P}(1) + \tilde{P}(2)$ and $p = P(1) / \theta$. Recall that

$$\psi(x) = \log\left(\frac{x}{1-x}\right) - \theta u x$$

for $x \in (0,1)$. It follows from (4.4) that

(A.3)
$$\psi(p) = v_{1m} - v_{2m} + (u_{12} - u_{22})\theta + \sum_{r=3}^{m-1} (u_{1r} - u_{2r})\tilde{P}(r).$$

We have that

$$\psi'(x) = \frac{1}{x(1-x)} - \theta u \le 4 - \theta u$$

so that with $\theta u < 4$, $\psi(x)$ is strictly increasing and only solution for *p* of (A.3) in (0,1) exists. If $\theta u > 4$, $\psi'(x)$ has two roots given by

$$\omega_1 = 0.5 - 0.5\sqrt{1 - \frac{4}{u\theta}}$$
 and $\omega_2 = 1 - \omega_1$

Note that by multiplying the nominator and denominator of

$$\frac{1+\sqrt{1-4/u\theta}}{1-\sqrt{1-4/u\theta}}$$

by $1 + \sqrt{1 - 4/u\theta}$, we obtain that

$$\frac{1+\sqrt{1-4/u\theta}}{1-\sqrt{1-4/u\theta}} = 0.5u\theta - 1 + 0.5u\theta\sqrt{1-4/u\theta}.$$

Hence, with f(x) given in (4.6) we get

$$\begin{split} \psi(\omega_2) + 0.5\theta u &= \log\left(\frac{1 + \sqrt{1 - 4/u\theta}}{1 - \sqrt{1 - 4/u\theta}}\right) - 0.5u\theta\sqrt{1 - 4/u\theta} \\ &= \log\left(0.5u\theta - 1 + 0.5u\theta\sqrt{1 - 4/u\theta}\right) - 0.5u\theta\sqrt{1 - 4/u\theta} \\ &= \log\left(0.5u\theta - 1 + 0.5\sqrt{u\theta(u\theta - 4)}\right) - 0.5\sqrt{u\theta(u\theta - 4)} = -f(u\theta). \end{split}$$

Similarly, it follows that

$$\psi(\omega_1) + 0.5\theta u = f(u\theta).$$

If $u\theta > 4$ and

(A.4)
$$\psi(\omega_1) + 0.5u\theta = f(u\theta) > v_{1m} - v_{2m} + (u_{12} - u_{22})\theta + \sum_{r=3}^{m} (u_{1r} - u_{2r})\tilde{P}(r)$$

(A.5)
$$\psi(\omega_2) + 0.5u\theta = -f(u\theta) < v_{1m} - v_{2m} + (u_{12} - u_{22})\theta + \sum_{r=3}^{m} (u_{1r} - u_{2r})\tilde{P}(r)$$

there exist at least three equilibria. It follows that (A.4) and (A.5) is equivalent to

$$f(\theta u) > |v_{1m} - v_{2m} + (u_{12} - u_{22})\theta + \sum_{r=3}^{m-1} (u_{1r} - u_{2r})\tilde{P}(r)|$$

which completes the proof.

Proof of Corollary 5:

When $\alpha_j = 0$ for all *j* it follows that there exist one equilibrium where P(j) = 1/m so that $\theta = 2/m$. Furthermore, since $\alpha_j - \alpha_k = 0$, the condition in Corollary 4 reduces to $\beta > m$, whence the result of the corollary follows. It follows from Corollary 1 and (4.4) that $P(1) = \theta p = p/m$ and $P(2) = \theta(1-p)$ where *p* is one of the three solutions to

$$\log\left(\frac{p}{1-p}\right) + 2\beta(0.5-p)/m = 0.$$

The last equation has three solutions when $\beta > m$, of which one equals 0.5.

Q.E.D.

Proof of Theorem 5:

From (3.4) it follows that an equilibrium (P,Q) is stable provided

(A.7)
$$u^{a}u^{b}L'(u^{a}Q+v_{12}^{a}-v_{22}^{a})L'(u^{b}P+v_{12}^{b}-v_{22}^{b})<1,$$

where $P = L(D_aQ + v_{12}^a - v_{22}^a)$. It follows that (A.7) is equivalent to

(A.8)
$$u^{a}u^{b}P(1-P)L(u^{b}P+v_{21}^{b}-v_{22}^{b})(1-L(u^{b}P+v_{21}^{b}-v_{22}^{b}))<1.$$

Assume first that $u^a u^b C_1 < 1$. Then by Theorem 2 and (A.8) a single equilibrium exists. Consider next the case when $u^a u^b C_1 > 1$. From (A.2) we see that g'(x), $x \in [0,1]$, can be both positive and negative in this case. The graph of g(x) is similar to the graph in Figure 1. In cases (ii) and (iii) of Theorem 2 we realize that when g'(P) > 0 at an equilibrium *P* then (A.8) holds whereas (A.8) does not hold when g'(P) < 0. Since $g'(P^B) < 0$ (Figure 1) P^B is unstable. In contrast, $g'(P^A) > 0$ and $g'(P^C) > 0$ are stable.

Q. E. D.

Proof of Corollary 7:

With m = 2 it follows from (5.2), with P(1) = P, that

$$P = H(P) = \frac{1}{1 + \exp(-\alpha_1 + \beta - 2\beta P)}$$

and

(A.9)
$$H'(P) = 2\beta H(P)(1 - H(P)) = 2\beta P(1 - P).$$

Hence, by Corollary 6 there exists a unique equilibrium P (say) which is stable when $\beta < 2$ because in this case H'(P) < 1. Consider next the cases (*ii*) or (*iii*) of Theorem 2. In this case (with ψ given by (3.6), $D = 2\beta$ and $v_{12} - v_{22} = \alpha_1 - \beta$) it follows that $\psi'(P) > 0$ which is equivalent to $2\beta P(1-P) < 1$ implying that the single equilibrium P is also stable in these cases. Consider finally case (iv) of Theorem 2. From Figure 1 we realize that at equilibrium B we have that $\psi'(P^B) < 0$, so that $2\beta P^B(1-P^B) > 1$ which shows that P^B is an unstable equilibrium. In a similar way it follows that the equilibria P^A and P^C are stable.

Proof of Theorem 3:

Recall that by (2.2) and (2.3) with $\lambda^a = \lambda^b = 1$ we have that

$$F_{j}^{a}(Q) = \frac{\exp(\sum_{r=1}^{n} v_{jr}^{a}Q(r))}{\sum_{s=1}^{m} \exp(\sum_{r=1}^{n} v_{sr}^{a}Q(r))}, \quad F_{k}^{b}(P) = \frac{\exp(\sum_{r=1}^{m} v_{rk}^{b}P(r))}{\sum_{s=1}^{n} \exp(\sum_{r=1}^{m} v_{rs}^{b}P(r))}$$

and $F(P,Q) = (F_1^a(Q), F_2^a(Q), ..., F_{m-1}^a(Q), F_1^b(P), F_2^b(P), ..., F_{n-1}^b(P))$. Let $x = (x_1, x_2, ...)$ be a vector in some Euclidian space and define the norm $\|\cdot\|$ by $\|x\| = \max_k |x_k|$. It follows that

(A.10)
$$\frac{\partial F_{j}^{a}(Q)}{\partial Q(r)} = P(j)(v_{jr}^{a} - v_{jn}^{a}) - P(j)\sum_{s} (v_{sr}^{a} - v_{sn}^{a})P(s)$$

and

(A.11)
$$\frac{\partial F_k^b(P)}{\partial P(r)} = Q(k)(v_{rk}^b - v_{mk}^b) - Q(k)\sum_s (v_{rs}^b - v_{ms}^b)Q(s).$$

Evidently, we have that

(A.12)
$$v_{jr}^{a} - v_{jn}^{a} - \sum_{s} (v_{sr}^{a} - v_{sn}^{a})P(s) \le (v_{jr}^{a} - v_{jn}^{a})(1 - P(j)) - \sum_{s \ne j} \min_{s} (v_{sr}^{a} - v_{sn}^{a})P(s)$$
$$= (v_{jr}^{a} - v_{jn}^{a} - \min_{s \ne j} (v_{sr}^{a} - v_{sn}^{a}))P(j)(1 - P(j)).$$

Similarly, we get that

(A.13)
$$v_{jr}^{a} - v_{jn}^{a} - \sum_{s} (v_{sr}^{a} - v_{sn}^{a}) P(s) \ge (v_{jr}^{a} - v_{jn}^{a} - \max_{s \ne j} (v_{sr}^{a} - v_{sn}^{a})) P(j)(1 - P(j)).$$

Thus, (A.12) and (A.13) implies that

(A.14)
$$|v_{jr}^{a} - v_{jn}^{a} - \sum_{s} (v_{sr}^{a} - v_{sn}^{a})P(s)| \le P(j)(1 - P(j))\max_{q} |v_{qr}^{a} - v_{qn}^{a} - \min_{s \ne q} (v_{sr}^{a} - v_{sn}^{a})|$$
$$= P(j)(1 - P(j))(\max_{s} (v_{sr}^{a} - v_{sn}^{a}) - \min_{s} (v_{sr}^{a} - v_{sn}^{a})).$$

Consequently, (A.10) and (A.14) yield

(A.15)
$$|\frac{\partial F_{j}^{a}(Q)}{\partial Q(r)}| \leq P(j)(1 - P(j))(\max_{s}(v_{sr}^{a} - v_{sn}^{a}) - \min_{s}(v_{sr}^{a} - v_{sn}^{a})).$$

In a similar way it follows that

(A.16)
$$|\frac{\partial F_k^b(P)}{\partial P(r)}| \le Q(k)(1 - P(k))(\max_s (v_{rs}^b - v_{ms}^b) - \min_s (v_{rs}^b - v_{ms}^b)).$$

Let P', P, Q' and Q be different vectors of choice probabilities. By the mean value theorem and (A.15) we therefore get that

(A.17)
$$|F_{j}^{a}(Q') - F_{j}^{a}(Q)| = |\sum_{r < n} \frac{\partial F_{j}^{a}(Q^{*})}{\partial Q_{r}} (Q'(r) - Q(r))| \le \max_{r < n} |Q'(r) - Q(r)|| \sum_{r < n} \frac{\partial F_{j}^{a}(Q^{*})}{\partial Q_{r}}|$$

 $\le \frac{1}{4} \max_{k < n} |Q'(k) - Q(k)| \sum_{r} [\max_{s} (v_{sr}^{a} - v_{sn}^{a}) - \min_{s} (v_{sr}^{a} - v_{sn}^{a})].$

Similarly, it follows from (A.16) that

$$(A.18)|F_k^b(P') - F_k^b(P)| \le \frac{1}{4} \max_{j < m} |P'(j) - P(j)| \sum_r [\max_s (v_{rs}^b - v_{ms}^b) - \min_s (v_{rs}^b - v_{ms}^b)].$$

Let *K* be a constant such that

(A.19)
$$\sum_{r} [\max_{s} (v_{sr}^{a} - v_{sn}^{a}) - \min_{s} (v_{sr}^{a} - v_{sn}^{a})] \le 4K$$

and

(A.20)
$$\sum_{r} [\max_{s} (v_{rs}^{b} - v_{ms}^{b}) - \min_{s} (v_{rs}^{b} - v_{ms}^{b})] \leq 4K.$$

Then it follows from (A.17), (A.18), (A.19) and (A.20) that

$$||F(P',Q') - F(P,Q))|| \le K ||(P',Q') - (P,Q)||.$$

Consequently, if K < 1 the mapping F(P,Q) is a contraction. Finally, remember that if a mapping is a contraction, it has a unique fixed point (Rudin, 1976). Hence, we have proved Theorem 1 when (i) and (ii) are satisfied.

Assume next that the vector norm is given by $||| x ||| = \sum_{k} |x_{k}|$. Then it follows similarly to the derivations above that

$$|F_{j}^{a}(Q') - F_{j}^{a}(Q)| \le P(j)(1 - P(j))[\max_{r}(\max_{s}(v_{sr}^{a} - v_{sn}^{a}) - \min_{s}(v_{sr}^{a} - v_{sn}^{a}))] |||Q' - Q|||$$

which implies that

(A.21)

$$||| F^{a}(Q') - F^{a}(Q) ||| \le ||| Q' - Q ||| [\max_{r} (\max_{s} (v_{sr}^{a} - v_{sn}^{a}) - \min_{s} (v_{sr}^{a} - v_{sn}^{a}))] \sum_{j < m} P(j)(1 - P(j))$$

$$\le ||| Q' - Q ||| [\max_{r} (\max_{s} (v_{sr}^{a} - v_{sn}^{a}) - \min_{s} (v_{sr}^{a} - v_{sn}^{a}))] \frac{m - 2}{m - 1}.$$

In a similar way we obtain that

(A.22)
$$||| F_k^b(P') - F_k^b(P) ||| \le ||| P'(j) - P(j) ||| [\max_r (\max_s (v_{rs}^b - v_{ms}^b) - \min_s (v_{rs}^b - v_{ms}^b))] \frac{n-2}{n-1}.$$

The result of Theorem 1 when (iii) and (iv) are satisfied now follows from (A.21) and (A.22).

Q. E. D.

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